Dynamic Maintenance of Monotone Dynamic Programs and Applications

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Uni Heidelberg, 26.04.2023

Making Dynamic Programming Dynamic

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Dynamic Programming (DP)

- Dynamic programming (DP) is a fundamental algorithm design paradigm
  - Complex problem is broken up into simpler subproblems, original problem is solved by combining the solutions
- Often prohibitively costly in practice
  - We often need time $\Omega(n^2)$ to compute solution
  - We even need $\Omega(n^2)$ space to store the DP table!

Speeding Up DPs

- Lots of different conditions that allow solving DPs more quickly
  - Total monotonicity, Monge property, certain convexity and concavity properties, Knuth–Yao quadrangle-inequality, …
- SMAWK algorithm computes solution for totally monotone DP tables in linear time $O(n + m)$
  - n = #rows, m = #columns
  - Naïve computation would take time $\Omega(mn)$

➡ This line of research focussed on static algorithms

Monge property

$$T_{i,j} + T_{i',j'} \leq T_{i,j'} + T_{i',j}$$

for all $i < i'$ and $j < j'$

12	21	38	76	89
47	14	14	29	60
21	8	20	10	71
68	16	29	15	76
97	8	12	2	6

Example of totally monotone matrix

example from https://courses.engr.illinois.edu/cs473/sp2016/notes/06-sparsedynprog.pdf

Dynamic DPs?

- **Dynamic algorithms:**
  Input is changing over time and we want to maintain a solution
  - *Goal:* Get small (polylogarithmic) update time
- Remember what I said about DPs earlier?
  - "Complex problem is broken up into simpler subproblems, original problem is solved by combining the solutions"

➡ Quite similar to how many dynamic algorithms are developed, so it should be "easy" to turn DPs into dynamic algorithms?

Question:
Is there a condition that implies that a DP can be made dynamic?

Problem: Arrays Do Not Work for Dynamic DPs

- **Problem:** When a single entry in the DP table changes, we often need to recompute $\Omega(m)$ entries — even in just a single row
    - If we store the DP as a two-dimensional array, this rules out the polylog update times that we hoped for

➡ Can we bypass this limitation by storing the DP table in a smarter way?

12	21	38	76	89
47	14	14	29	60
21	8	20	~~10~~	~~71~~
68	16	~~29~~	~~15~~	~~76~~
97	~~8~~	~~12~~	~~2~~	~~6~~
	9	14	32	64

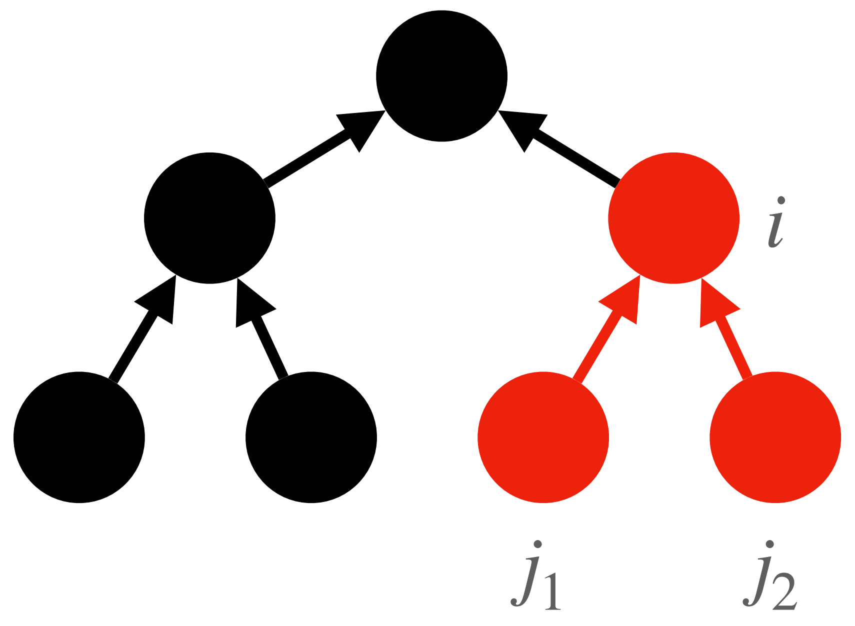
Our Results

Answer:
Yes! There is a condition that implies that
a DP can be made dynamic!

If the rows are monotonically increasing
and we allow approximation*,
then we can store the DP table in a smarter way
and get polylog update times.

*… and a few more technical conditions apply

Notation



- Consider an $n \times m$ DP table T with entries in $[0, W]$
- The rows are monotone if $T_{i,j} \leq T_{i,j+1}$ for all i and j
- We say that a DP table $\tilde{T}$ is an α-approximation of T if $T_{i,j} \leq \tilde{T}_{i,j} \leq \alpha \cdot T_{i,j}$ for all i and j
- We assume the DP can be computed row-by-row
- Dependency tree for the DP: tree which encodes whether computing row i requires solution for row j

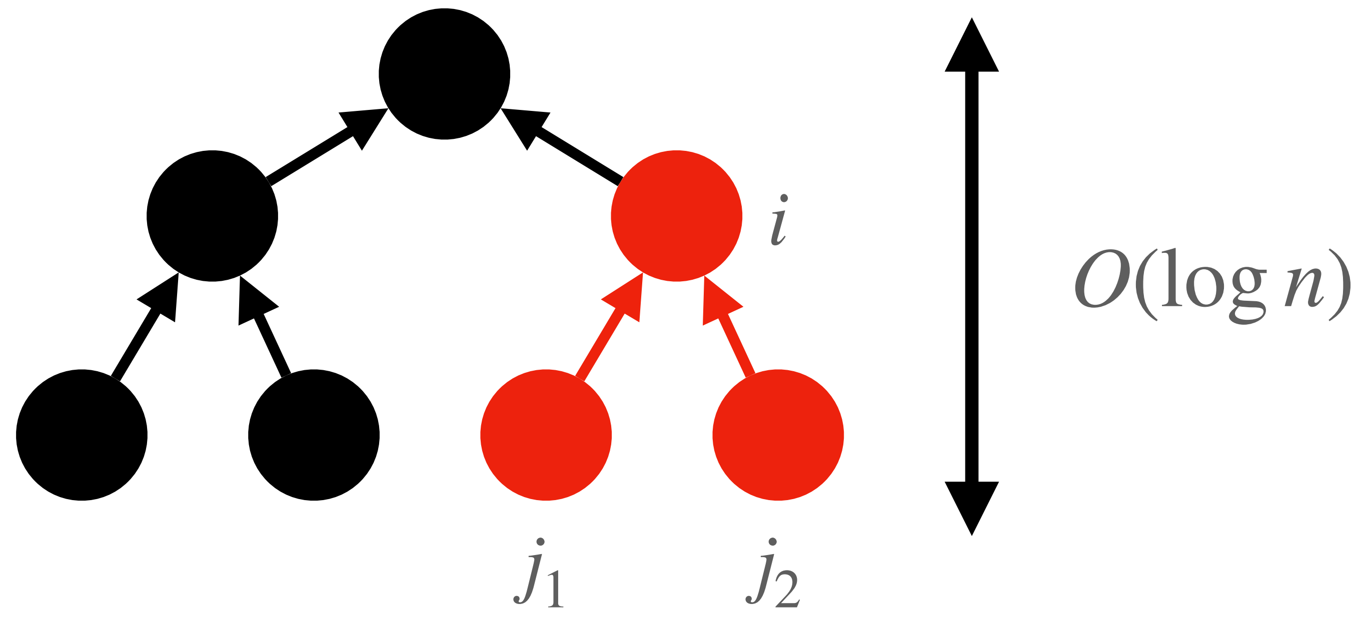
0	4	5	10	17	
0	0	14	29	60	
0	1	1	2	5	j_1
0	16	29	29	29	j_2
0	8	18	30	31	i

0	4.2	5.1	10	17.1
0	0	14	30	61
0	1.1	1.1	2.1	5.5
0	16.3	30	30	30
0	8.1	18.2	30	31.3

General Framework to Make DPs Dynamic

- **Assumption:** DP has n monotone rows and dependency tree of height $O(\log n)$ and rows are "easy to compute"
- **Static result:** We can compute a $(1+\varepsilon)$-approximation of the DP table in near-linear time and space $\tilde{O}(n)$
  - Every entry is correct up to a multiplicative $(1+\varepsilon)$-factor
  - Much more efficient than writing down the entire table as an array in time $\Omega(mn)$
- **Dynamic result:** When entries in the DP table change, we can update the *entire* table in polylogarithmic time $\tilde{O}(1)$

0	4	5	10	17
0	0	14	29	60
0	1	1	2	5
0	16	29	29	29
0	8	12	20	22



Why Should DPs Have Monotone Rows?

- For many optimization problems, the columns correspond to budget constraints (e.g., Knapsack)
  - k'th column = "Maximum objective function with budget *at most* k"
  - Monotone rows appear automatically
- Sometimes we have exact budget constraints ("with budget *exactly* k")
  - Often the DP can be adapted such that it works in the "budget at most k"-setting
  - In the paper, we do this for k-Balanced Graph Partitioning
- Sometimes other tricks can help
  - For simultaneous source location, a DP by Andreev et al. did not fit our framework (e.g., used negative values)
  - In the paper, we consider the "inverse" of this DP — takes only positive values, fits into our framework

Balanced Graph Partitioning

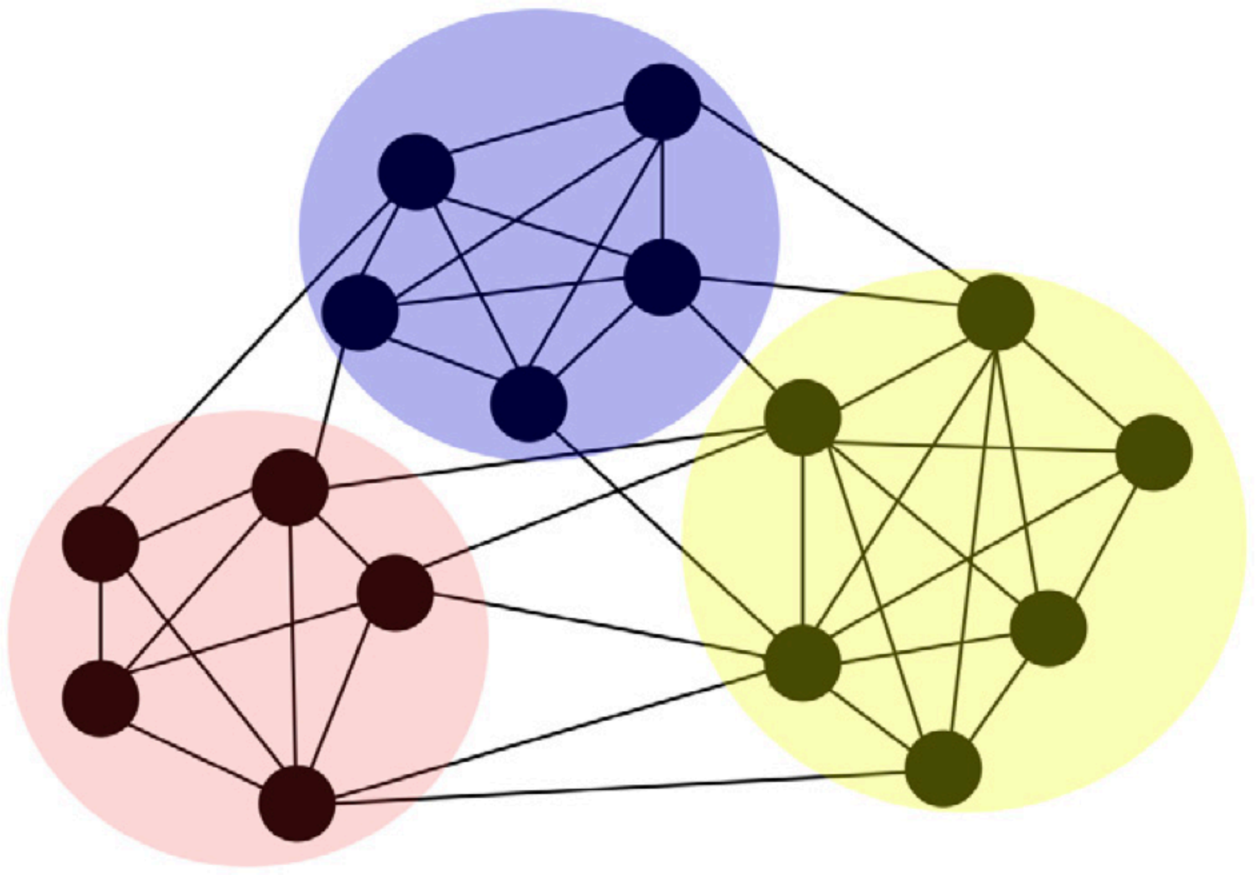
- **Problem:**
  Given a graph $G = (V, E, w)$ partition the vertices into k groups $V_1, \ldots, V_k$, such that

  each group V_i contains n/k vertices and

  $$\mathrm{cut}(V_1, \ldots, V_k) = \sum_{(u,v)\in E:\ u\in V_i, v\in V_j} w(u, v)$$ is minimized

- Important pre-processing step in many distributed graph algorithms, popular heuristics METIS have thousands of citations and everyone loves KaHIP

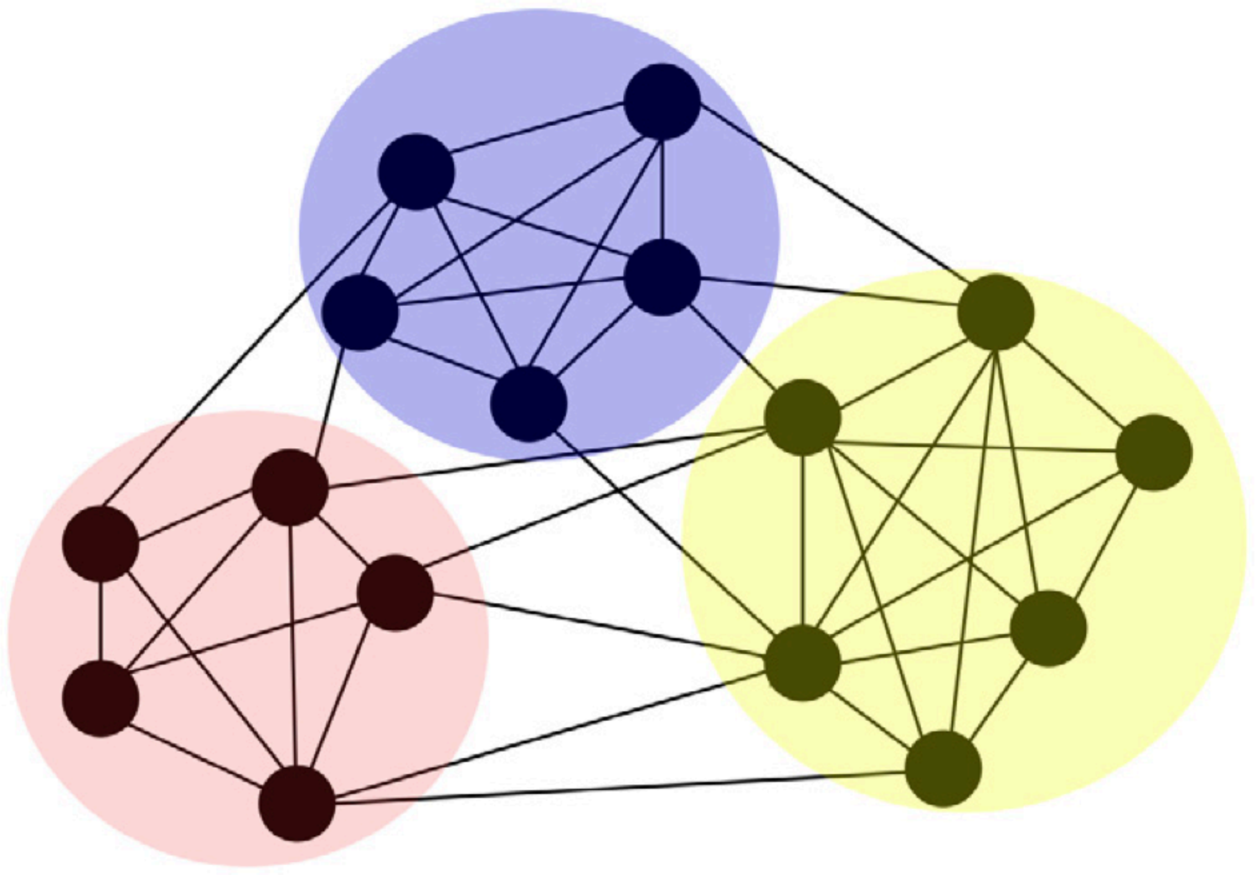
- **Bicriteria version:** Each group may contain up to $(1 + \varepsilon)n/k$ vertices, we compare cut-value against optimal solution that has to satisfy the constraint exactly



Picture taken from Rais et al., https://doi.org/10.20965/jaciii.2019.p0005

Balanced Graph Partitioning

- **Our results:**
  - First static near-linear time algorithm computing a bicriteria $O(\lg^4 n)$-approximation
    - Best polynomial-time algorithm computes a bicriteria $O(\lg^{1.5} n \lg \lg n)$-approximation in time $\Omega(n^4)$ (Feldmann and Foschini, 2015)
  - First dynamic algorithm with subpolynomial update time for unweighted graphs which maintains a bicriteria $n^{o(1)}$-approximation under edge insertions and deletions (update time $O_{k,\varepsilon}(1) \cdot n^{o(1)}$)
  - We simplify and generalize the DP by Feldmann and Foschini



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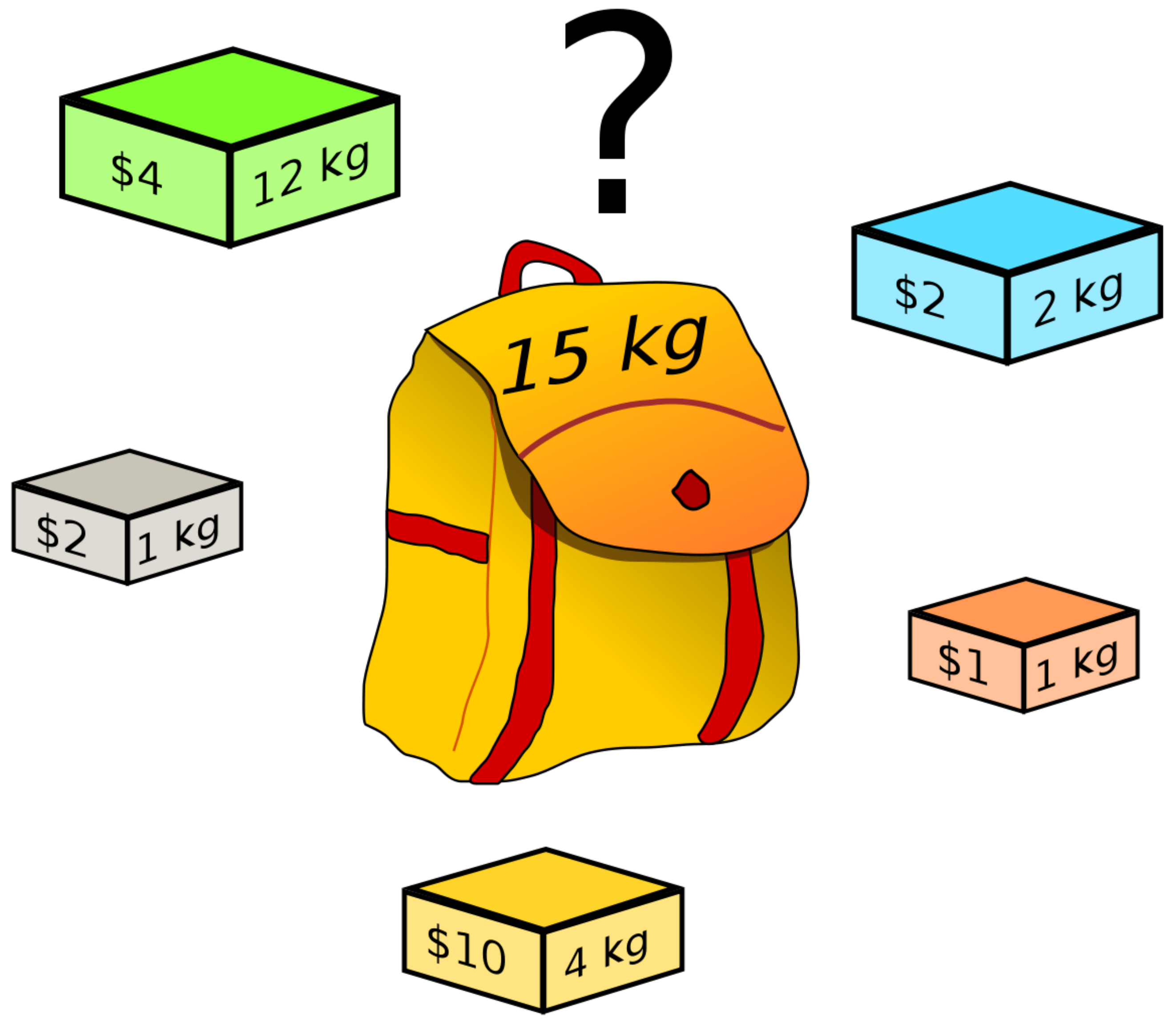
Fully Dynamic Knapsack

- **Problem:**
  Given a budget B and n items with profits $v_1, \ldots, v_n$ and weights $w_1, \ldots, w_n$, maximize $\sum_{i \in I} v_i$ such that $\sum_{i \in I} w_i \leq B$

- **Dynamic version:** Items are inserted and deleted

- **Our result:** Maintain a $(1 + \varepsilon)$-approximation with update time $O(\varepsilon^{-2} \log^2(nW))$
  - Improves upon Eberle et al. (2021) who obtained update time $O(\varepsilon^{-9} \log^4(nW))$
  - Can you implement this result?



Picture from Wikipedia
https://commons.wikimedia.org/wiki/File:Knapsack.svg

Further Results

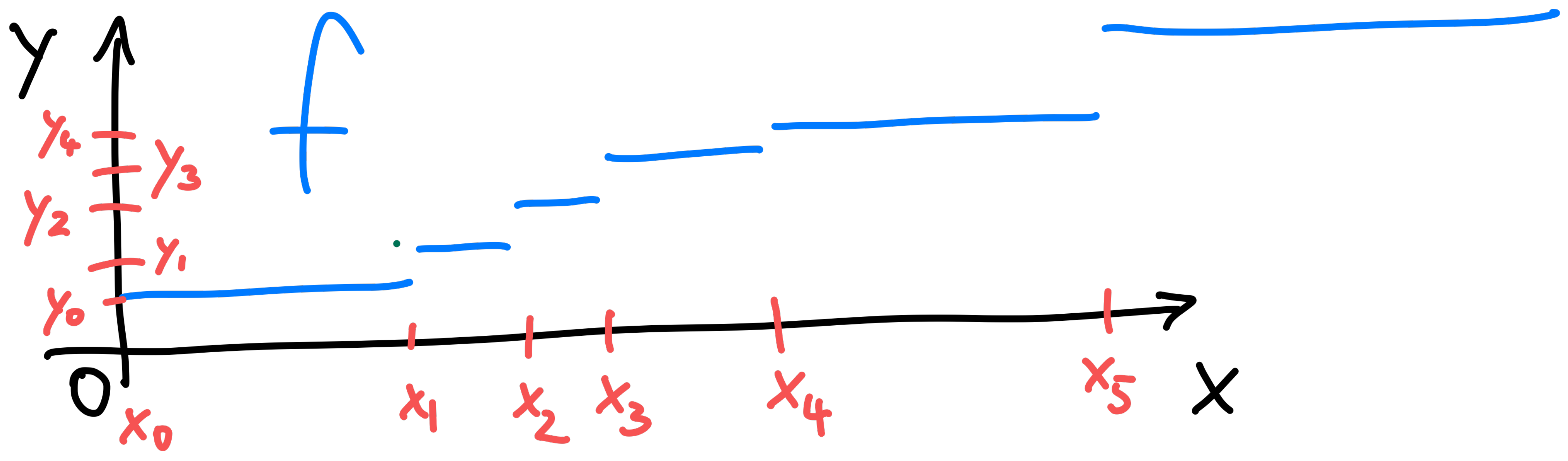
- Two tricks to make rows of DPs monotone
- Lower bounds for Dynamic k-Balanced Graph Partitioning and Dynamic Knapsack showing that if an algorithm only stores a single solution then update time is high
  - Suggests that DP-style implicit solutions are inevitable
- For simultaneous source location: First near-linear time static algorithm and first dynamic algorithm with subpolynomial update time
- First dynamic algorithm for ℓ_∞-Necklace with additive approximation $\pm\varepsilon$ and update time $O(\varepsilon^{-2})$

How do we get these results?

Store the DP Table's Monotone Rows using Piecewise Constant Functions

What is a Piecewise Constant Function?

- A piecewise constant function $f\colon [0,t] \to [1,W]$ looks like this:



- The set of tuples $(x_1, y_1), \ldots, (x_p, y_p)$ encodes the x-coordinates and y-coordinates
- New complexity measure: number of pieces p of the function
- *Interpretation:* For each row i of the DP table T, we store a function f_i such that $f_i(j) = T_{i,j}$

➡ Looking at the function f_i reveals the entire i'th row

Efficient Operations

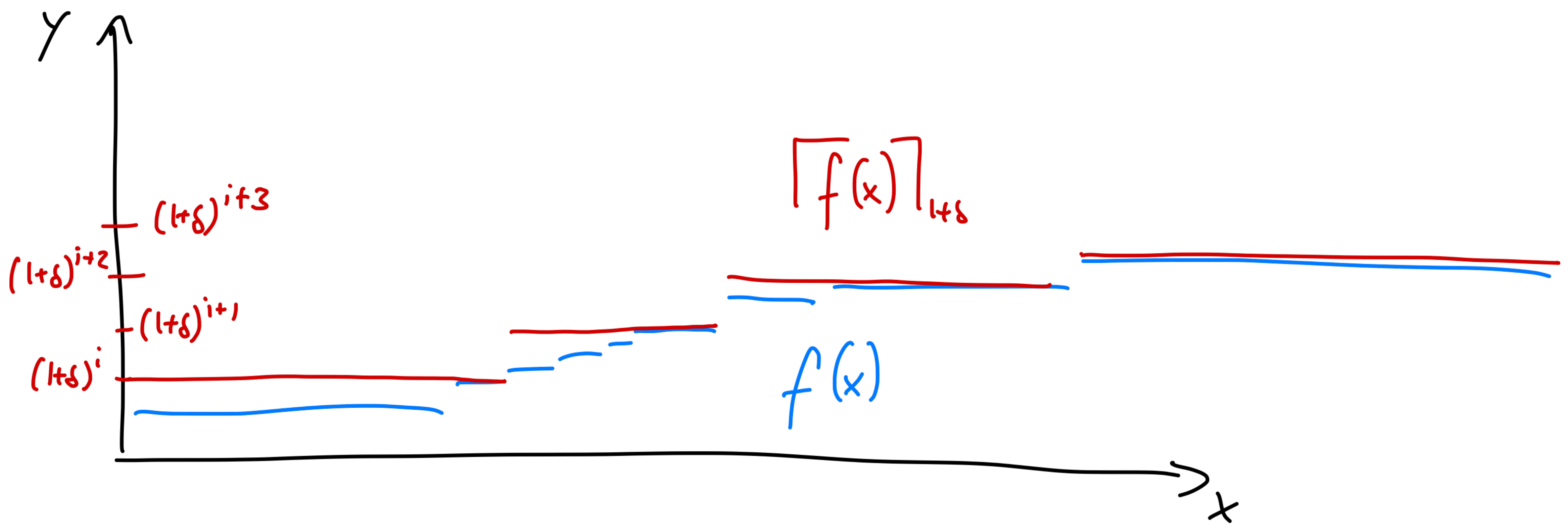
- Let $g, h\colon [0,t] \to [0,W]$ be piecewise constant functions with at most p pieces.
  Then we can compute:
  - $f_{\min}$ with $f_{\min}(x) := \min\{g(x), h(x)\}$ in time $\tilde{O}(p)$ and at most $2p$ pieces
  - f_{add} with $f_{\text{add}}(x) := g(x) + h(x)$ in time $\tilde{O}(p)$ and at most $2p$ pieces
  - the $(\max, +)$-convolution $f_{\text{conv}} = g \oplus h$ of g and h in time $\tilde{O}(p^2)$ and with $\tilde{O}(p^2)$ pieces, where
  $$f_{\text{conv}}(x) = \max_{x' \in [0,x]} g(x') + h(x - x')$$

➡ We compute the *entire* functions at once

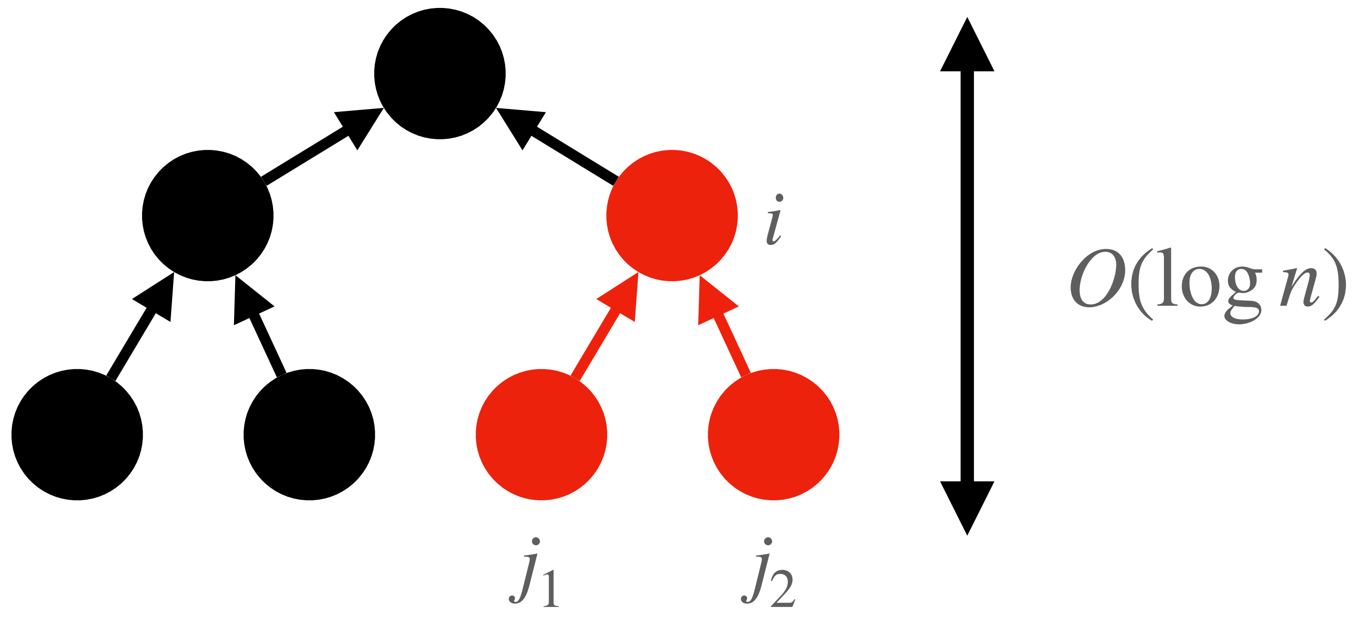
➡ Running times are fast if p is small, no dependency on the size of the domain $[0,t]$!

Ensuring Few Pieces

- How do we ensure that the number of pieces stays small?
- We round $f(x)$ to powers of $1 + \delta$: $\lceil f(x) \rceil_{1+\delta} = \min\{(1+\delta)^i : (1+\delta)^i \geq f(x), i \in \mathbb{N}\}$
  - If $f(x) \in [1, W]$ for all x, then $\lceil f(x) \rceil_{1+\delta}$ only takes $O(\log_{1+\delta}(W))$ values
  - If f is monotone, we have ≤1 piece for each value and thus $\lceil f \rceil_{1+\delta}$ has at most $O(\log_{1+\delta}(W))$ pieces
  - We can perform all operations from before in time $\log_{1+\delta}^{O(1)}(W)$

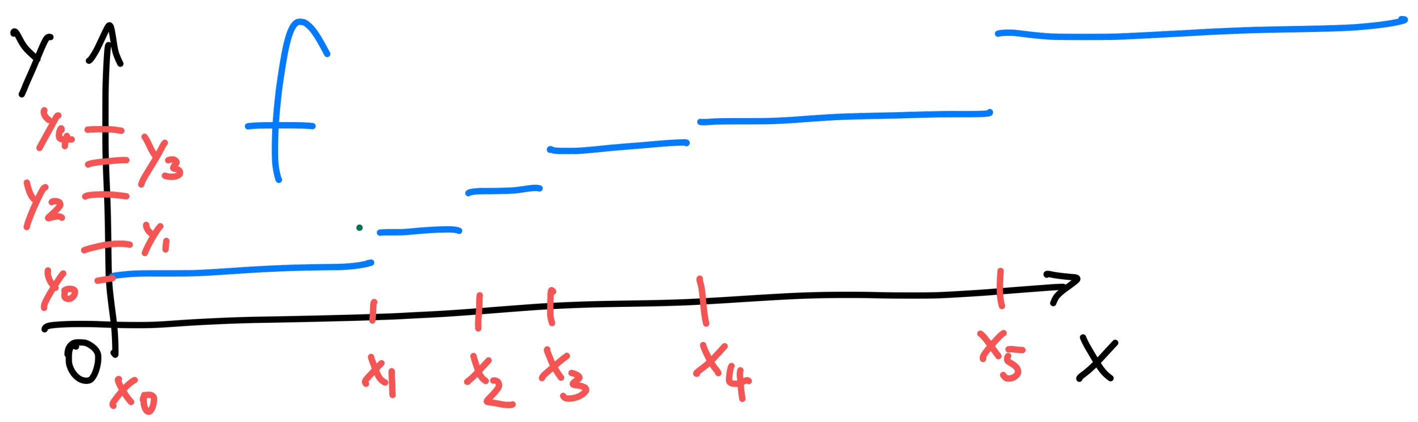


Why is this Useful in DPs?



- We store the rows i of the DP table as piecewise constant functions f_i
- We compute entire rows in polylogarithmic time, using operations for our piecewise constant functions (rather than computing them entry-by-entry)
- **Recall our assumption:** DP has n monotone rows and dependency tree of height $O(\log n)$ and rows are "easy to compute"
  - "easy to compute": to compute a row, we use only $O(1)$ operations of type $\min\{g, h\}$ and $g + h$, and at most one $(\max, +)$-convolution
  - After computing a row, we perform a rounding step $\lceil f \rceil_{1+\delta}$
    - Bounds the number of pieces, allows us to compute each row in time $\tilde{O}(1)$
    - Since the dependency tree has small height, error does not compound too much

0	4	5	10	17
0	0	14	29	60
0	1	1	2	5
0	16	29	29	29
0	8	12	20	22



Application:
Fully Dynamic Knapsack

Our main technical contribution in the paper is for k-Balanced Graph Partitioning.
But the result for fully dynamic Knapsack is very illustrative for our approach.

Fully Dynamic Knapsack

- **Problem:**
  Given a budget B and n items with profit $v_1, \ldots, v_n$ and weights $w_1, \ldots, w_n$, maximize $\sum_{i \in I} v_i$ such that $\sum_{i \in I} w_i \leq B$

- **Dynamic version:** Items are inserted and deleted

- **Our result:** Maintain a $(1 + \varepsilon)$-approximation with update time $O(\varepsilon^{-2} \log^2(nW))$

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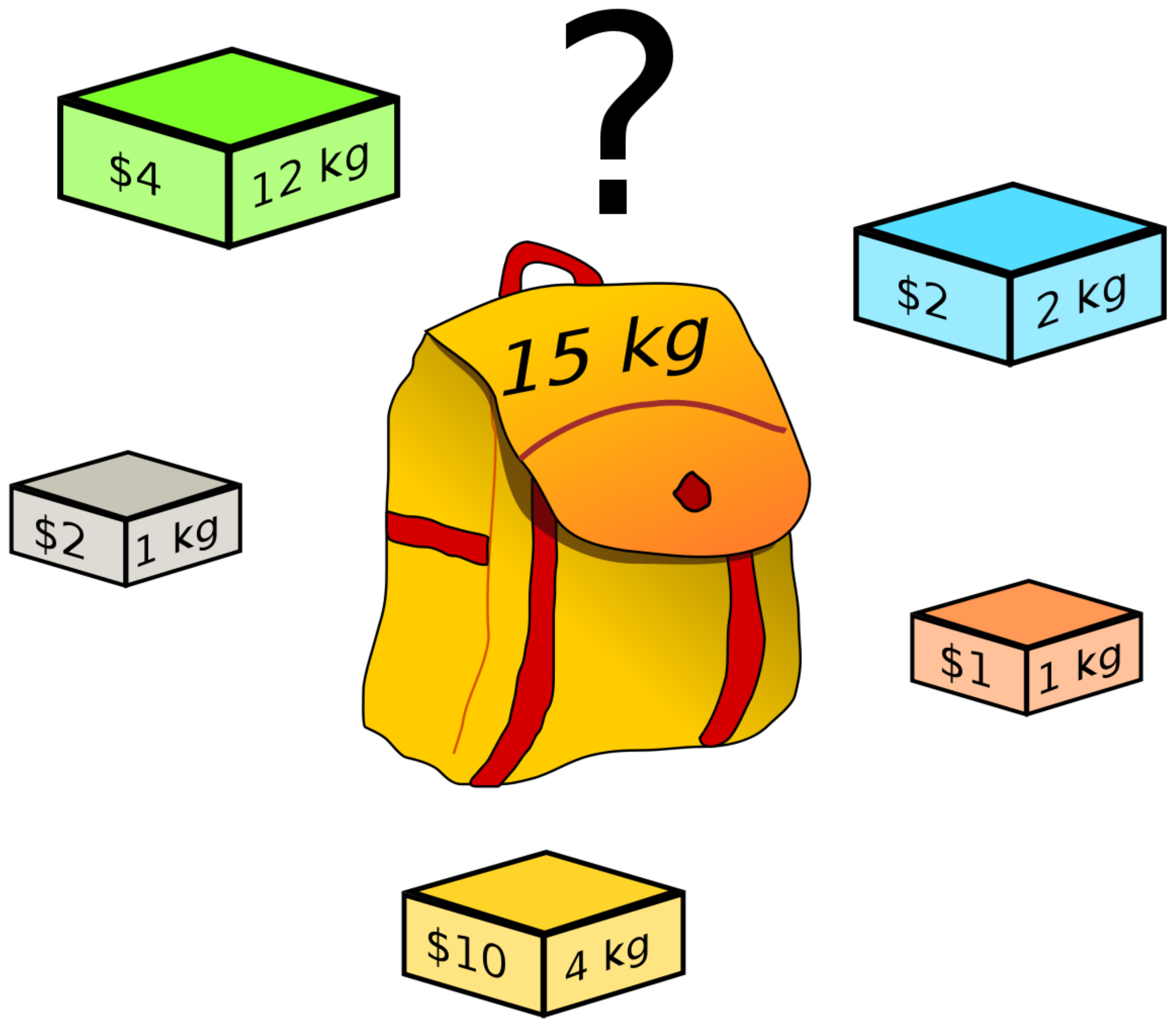
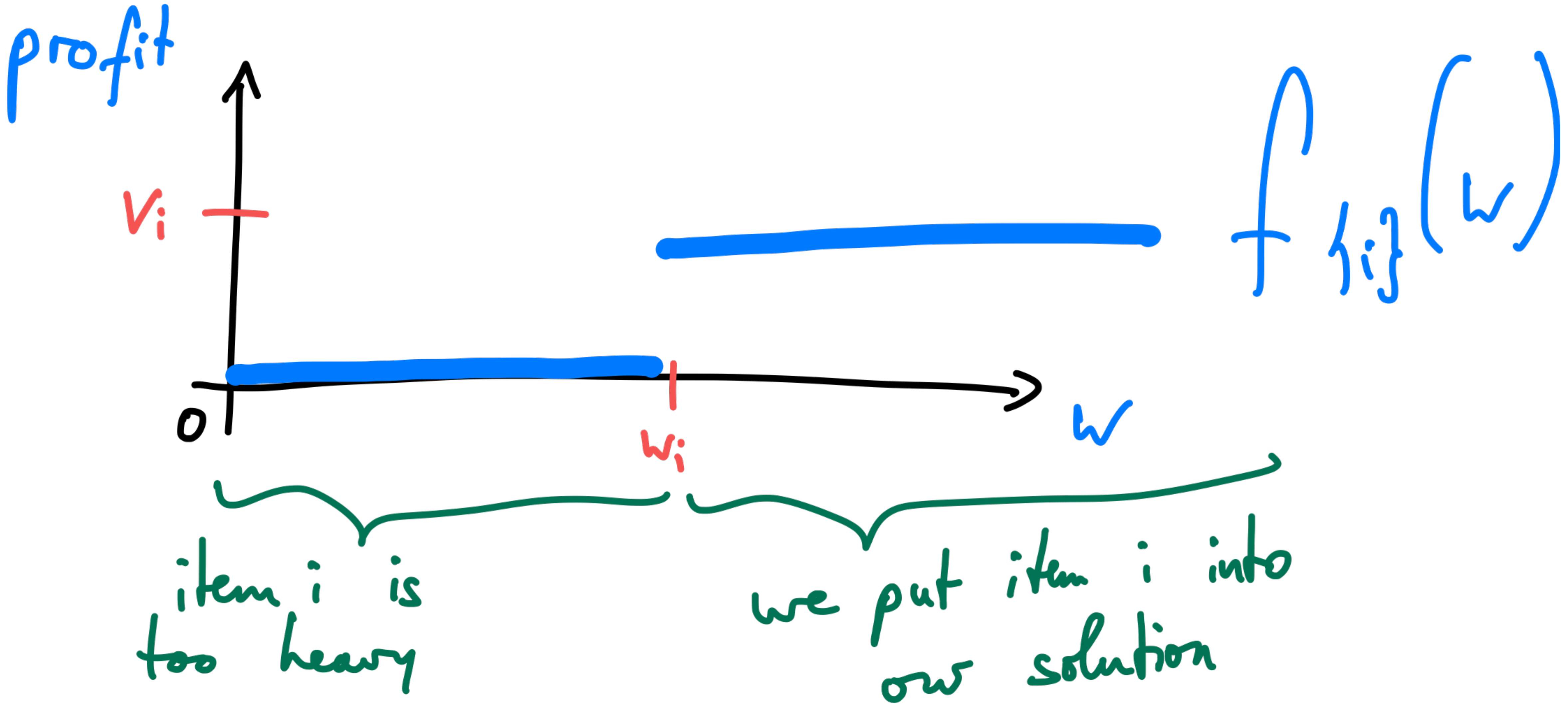


Picture from Wikipedia
https://commons.wikimedia.org/wiki/File:Knapsack.svg

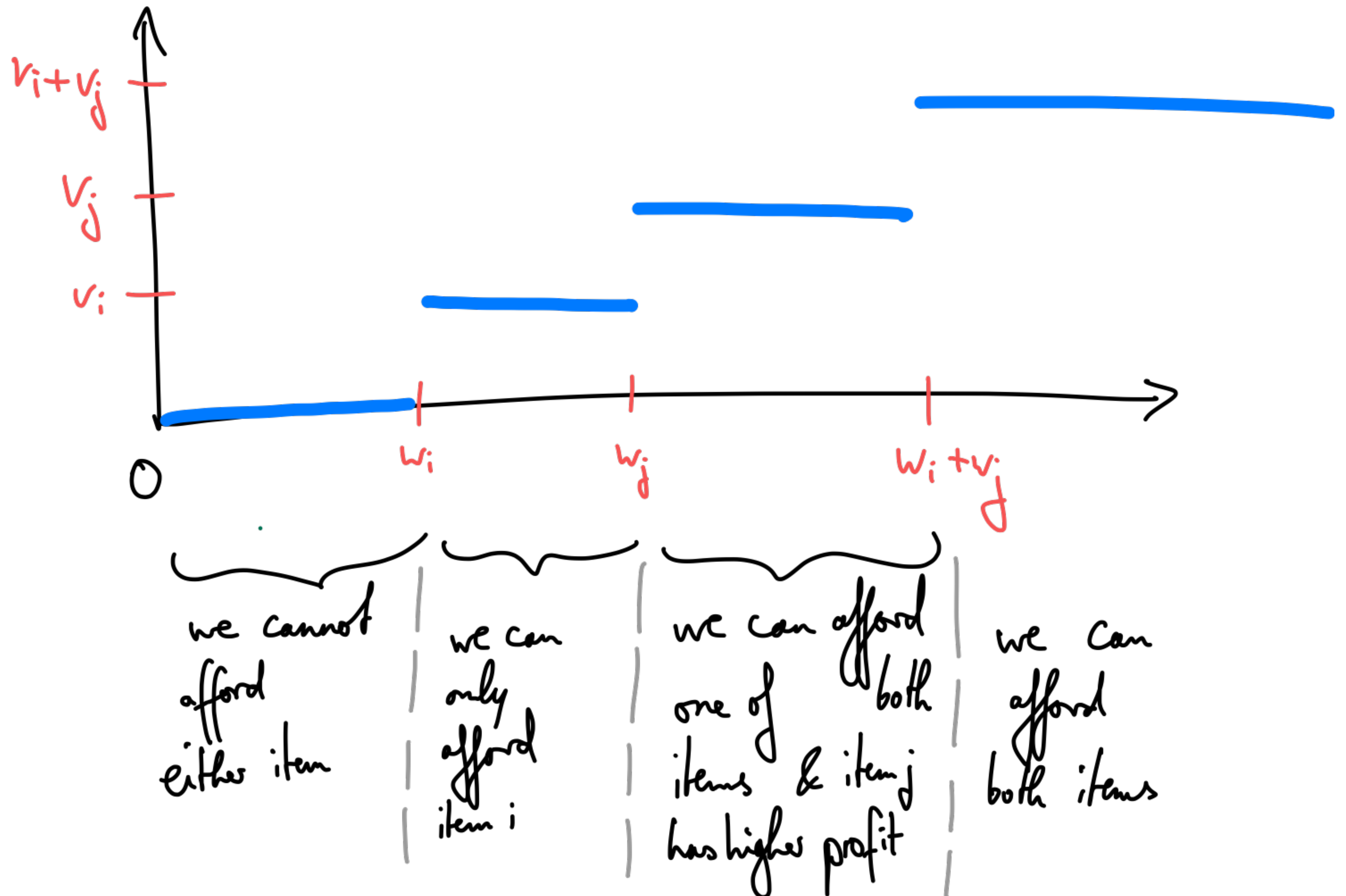
Warm-Up: Existing Algorithm

- How can we solve Knapsack using piecewise constant functions?
- Consider an item i with profit v_i and weight w_i
- Set $f_{\{i\}}(w)$ = maximum profit if we can spend weight w and can only use items in the set $\{i\}$



Two Items

- Suppose now we have two items i and j such that $v_i \leq v_j$ and $w_i \leq w_j$
- Set $f_{\{i,j\}}(w) =$ maximum profit if we can spend weight w and can only use items in the set $\{i, j\}$
- Then $f_{\{i,j\}}(w)$ looks like this:



How to Compute $f_{\{i,j\}}(w)$?

- Observe that $f_{\{i,j\}}(w)$ can be computed via a $(\max, +)$-convolution

$$f_{\{i,j\}}(w) = \max_{w_1 \in [0, w]} f_{\{i\}}(w_1) + f_{\{j\}}(w - w_1)$$

we have budget w

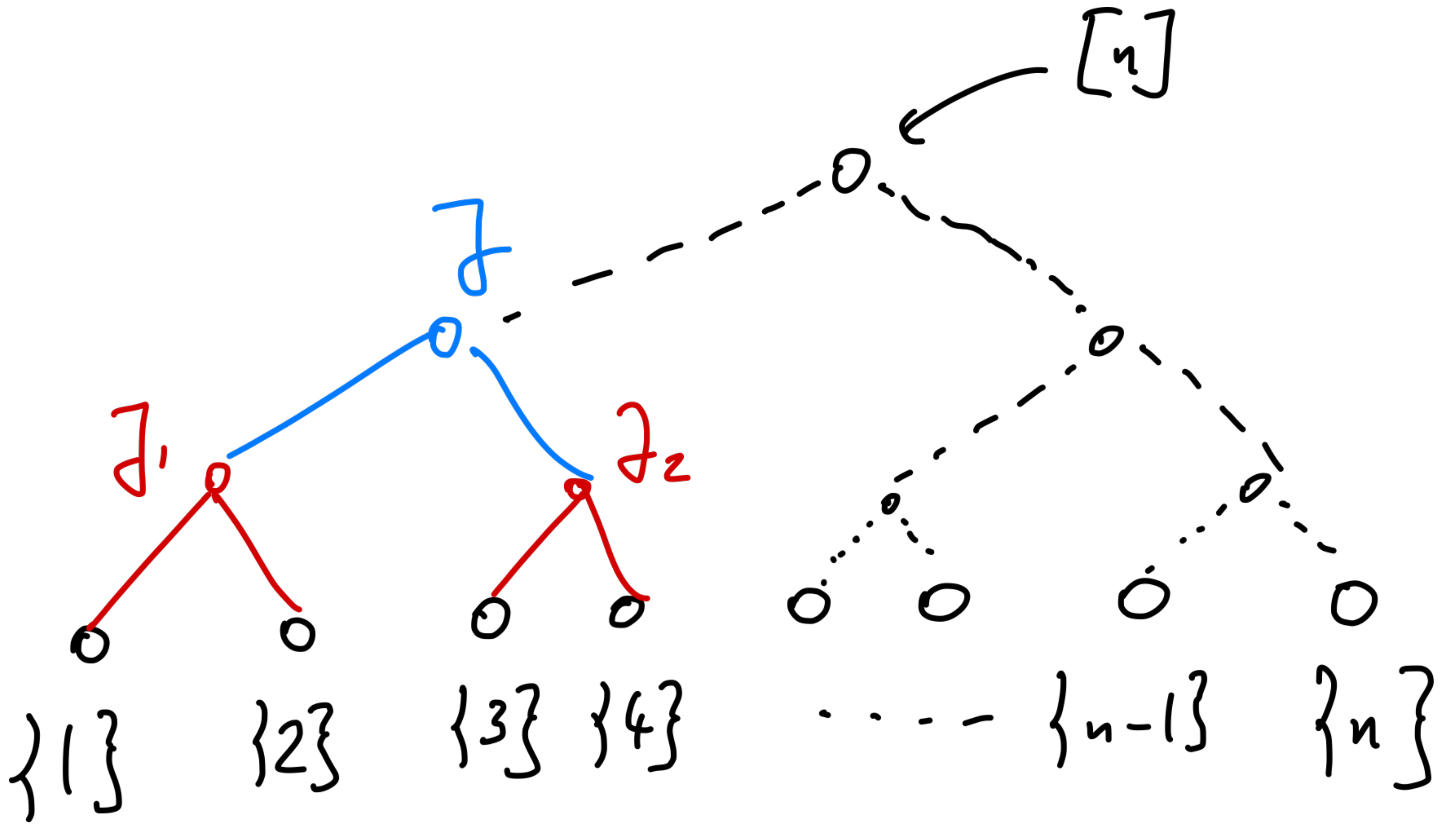
maximize over all possible budget allocations

use budget w_1 on item i

use the remaining budget $w - w_1$ on item j

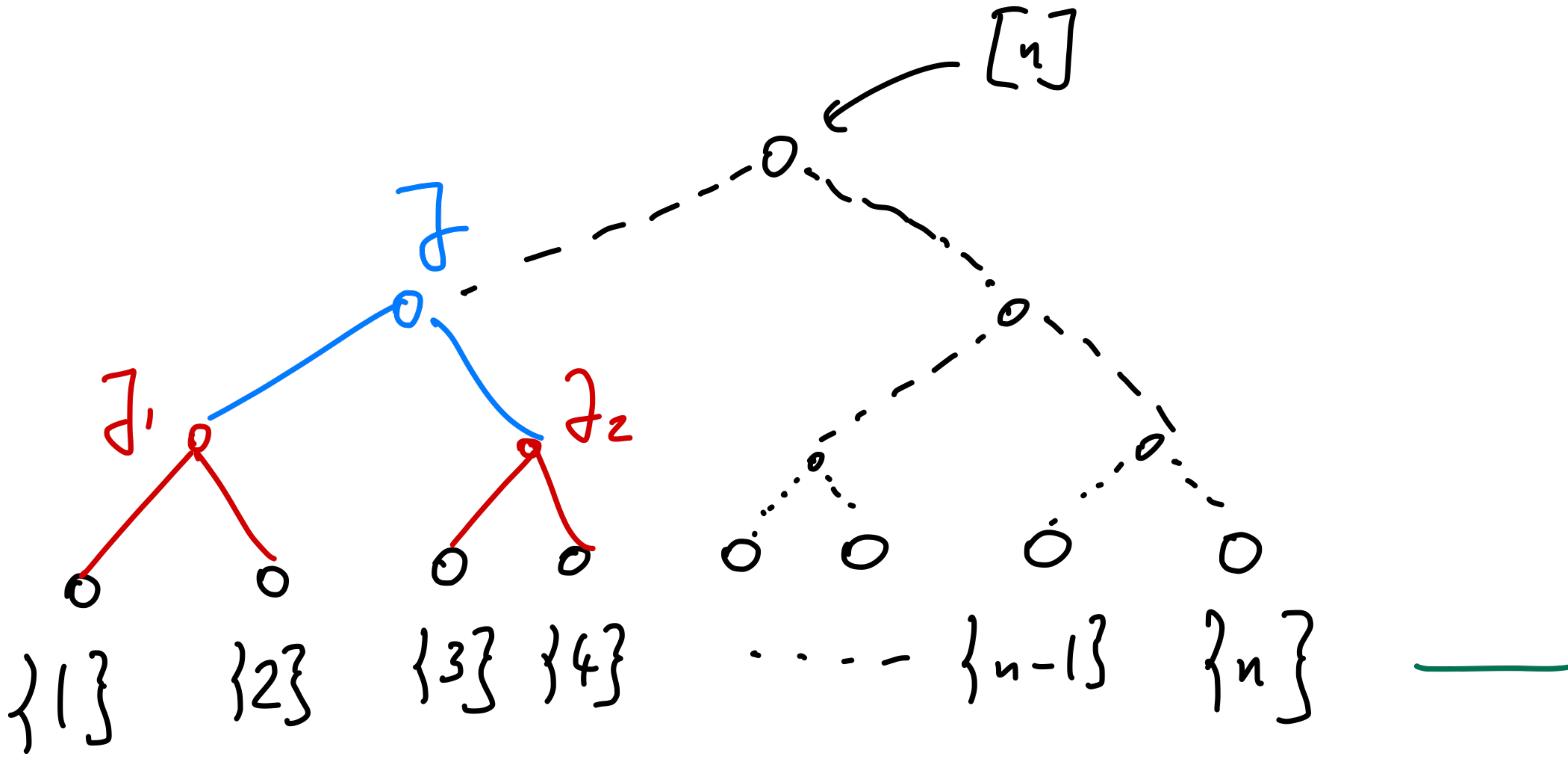
General Case of More Than Two Items

- More generally, set $f_J(w) =$ maximum profit if we can spend weight w and can only use items in the set J
    - $f_{[n]}(B)$ is the optimal solution for the global problem
- If $J = J_1 \dot{\cup} J_2$ then $f_J(w) = \max_{0 \le w' \le w} f_{J_1}(w') + f_{J_2}(w - w')$
- **Algorithm:**
    - Compute the DP bottom-up
    - In each internal node, we compute f_J as $(\max, +)$-convolution of f_{J_1} and f_{J_2}
    - Then we set $f_J = \lceil f_J \rceil_{1+\delta}$



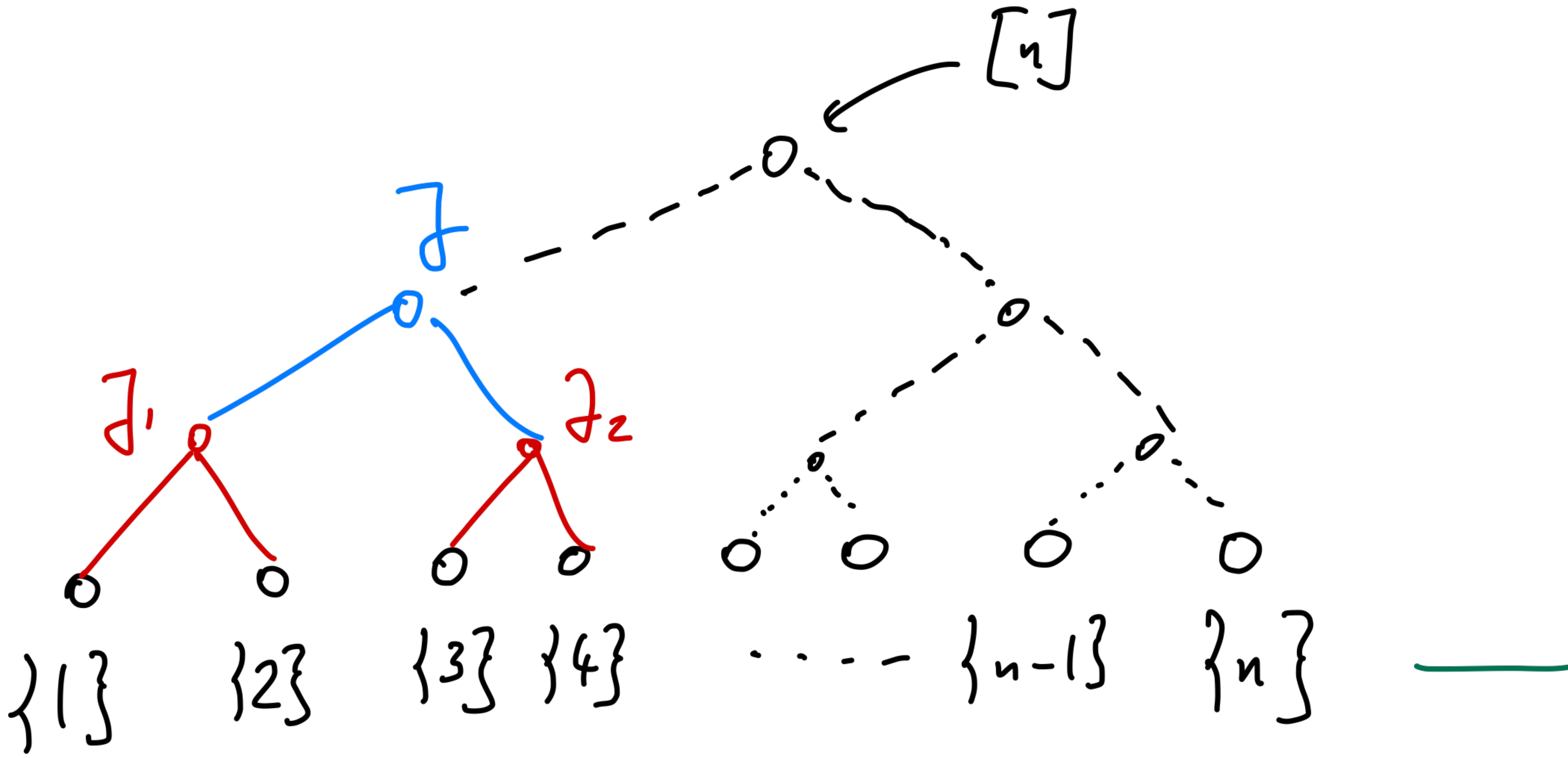
Analysis

- **Algorithm:**
  - In each internal node we compute f_J as $(\max, +)$-convolution of f_{J_1} and f_{J_2}
  - Then we set $f_J = \lceil f_J \rceil_{1+\delta}$
- **Approximation ratio:**
  - At each level, we lose approximation factor $1 + \delta$ because of rounding
  - Since the dependency tree has height $O(\log n)$, our approximation ratio is $(1+\delta)^{O(\log n)} \leq \exp(\delta \cdot O(\log n)) \leq 1 + \varepsilon$ for $\delta = \log(1+\varepsilon)/O(\log n)$
- **Running time:**
  - Each function has at most $O(\log_{1+\delta}(W))$ pieces, thus convolution takes time $O(\log^2_{1+\delta}(W))$
  - Using δ as above, total time is $O(n \cdot \varepsilon^{-2} \log^2(W) \log^2(n))$



Dynamic Knapsack

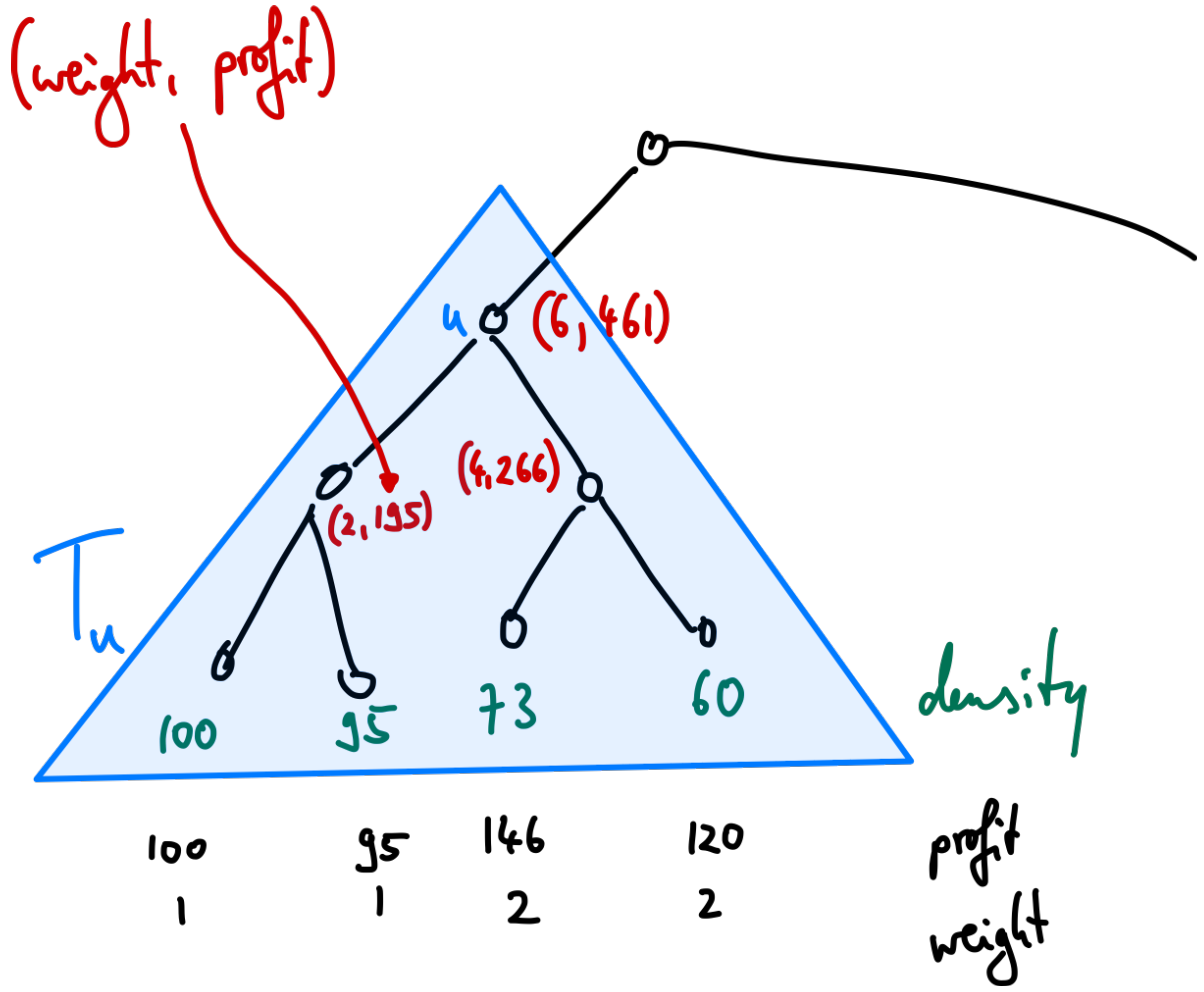
- **Dynamic version:**
  - Suppose we can change item profits and weights
  - *Update*(i, v, w): set $v_i = v$ and $w_i = w$
  - After update for item i, recompute leaf–root path from node i to root
  - Takes update time $O(\varepsilon^{-2} \log^2(W) \log^3(n))$
- But we can be even faster: update time $O(\varepsilon^{-2} \log^2(nW))$



Even Faster
Fully Dynamic Knapsack

Excursion: Fully Dynamic *Fractional* Knapsack

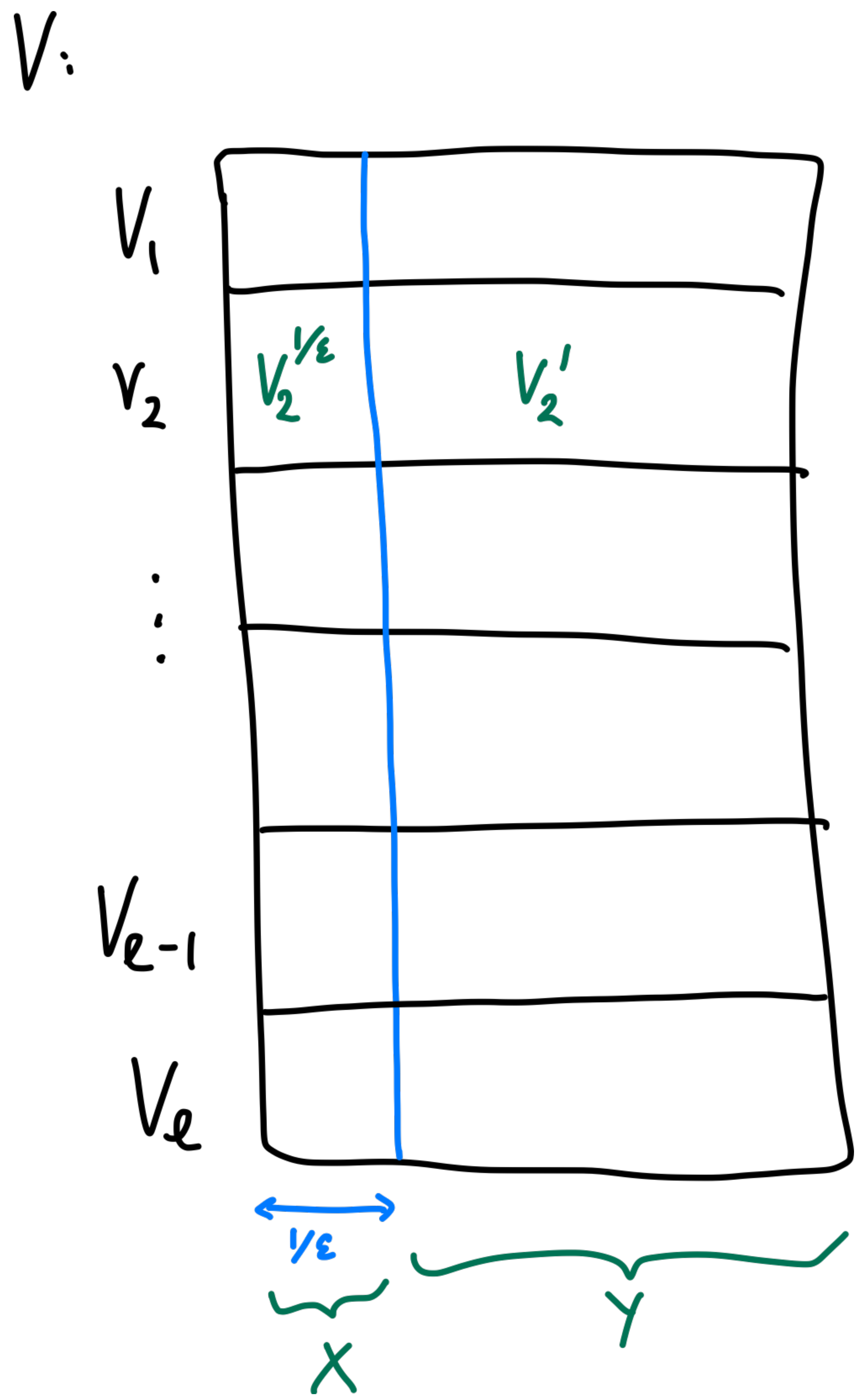
- Fractional Knapsack:
  - You are allowed to add items *fractionally* to your solution
  - *Example:*
    Suppose we have budget $B = 10$ and two items with weights 1000 and 10, and profits 500 and 1, resp.
    You can add a 0.01-fraction of the item with weight 1000 to get profit 5.
- Computing an exact solution:
  - Order items by their *density* $\frac{v_i}{w_i}$ and keep on adding items until budget B is used up.
  - If necessary, "cut" the final item fractionally.
- Dynamic version:
  - Use binary search tree to maintain order of items, based on their density
  - In each internal node u of the search tree,
    store the total weight and the total profit of all items in that subtree T_u



Faster Dynamic Knapsack

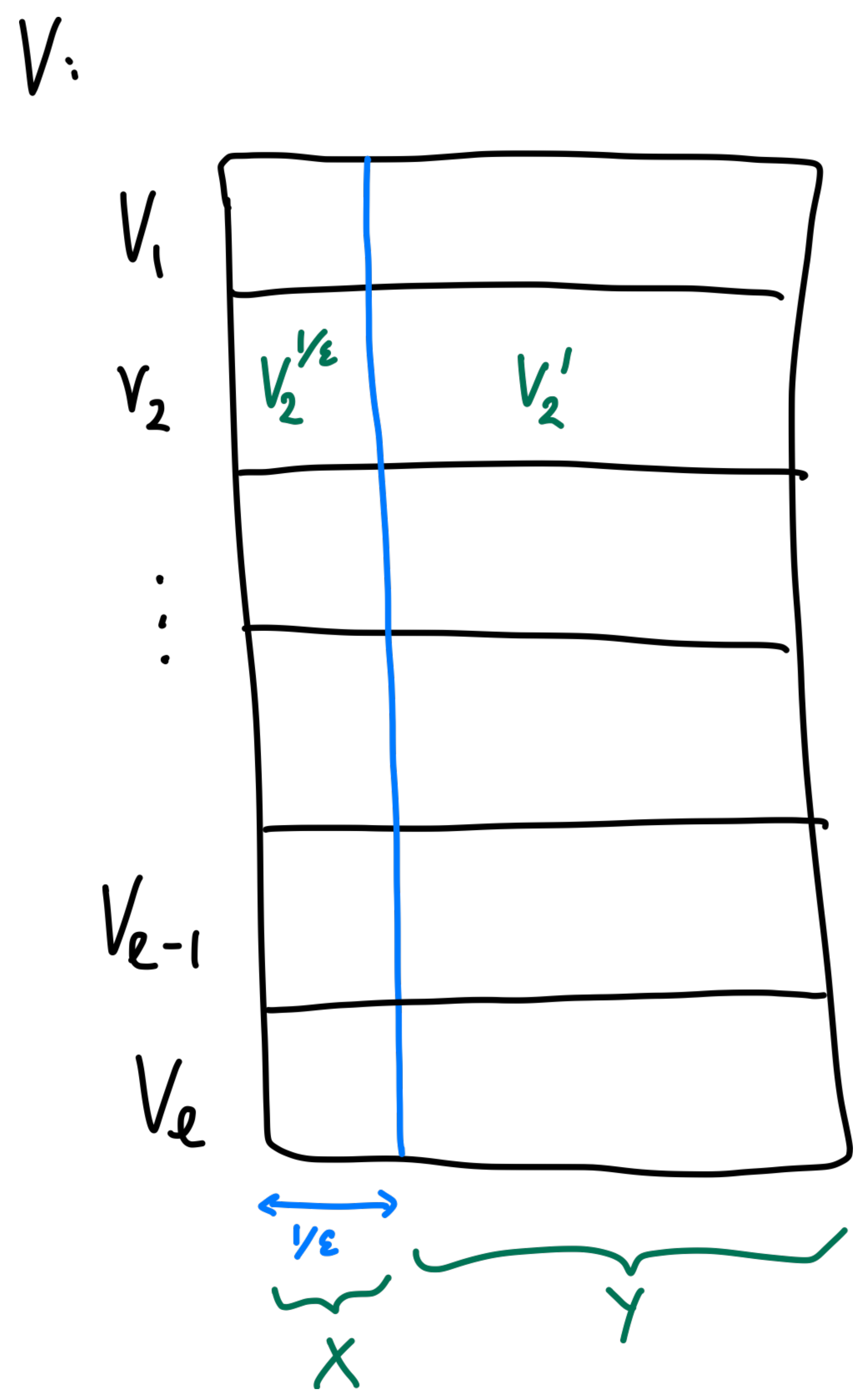
- Partition the items into profit classes $V_\ell = \{i\colon (1+\varepsilon)^\ell \le v_i < (1+\varepsilon)^{\ell+1}\}$
- Set $V_\ell^{1/\varepsilon}$ to the $1/\varepsilon$ items from V_ℓ of smallest weight, $V'_\ell = V_\ell \backslash V_\ell^{1/\varepsilon}$
- Consider the $1/\varepsilon$ items $X = \bigcup_{\ell \ge 0} V_\ell^{1/\varepsilon}$ of smallest weight from each class,

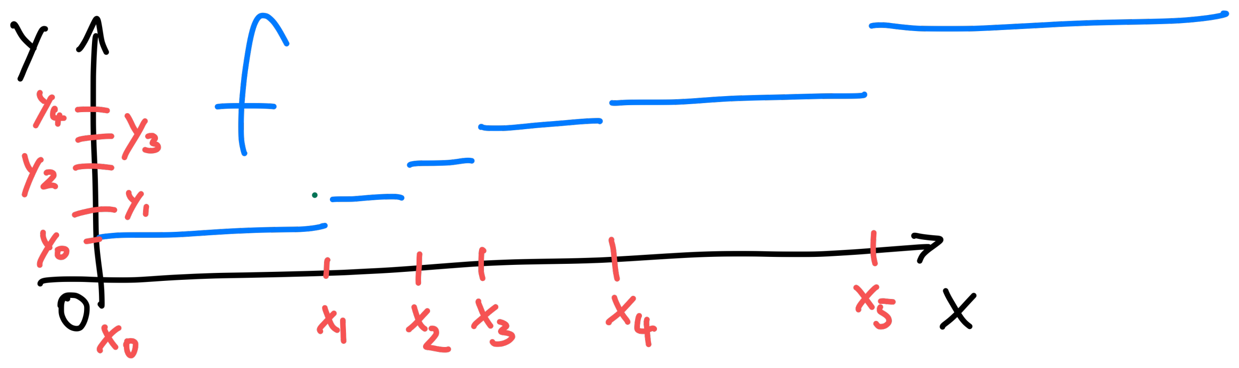
  and the other items $Y = \bigcup_{\ell \ge 0} V'_\ell$
- Maintain Y in a binary search tree, sorted by density v_i/w_i (as in last slide)
- Note that $|X| = \ell \cdot 1/\varepsilon = O(\varepsilon^{-2}\log(W))$
  - Maintain X using our data structure from before, since $|X|$ is small, update time is $O(\varepsilon^{-2}\log^2(nW))$



Answering Queries

- We maintain X using our data structure from before
  - Solution is stored as a piecewise constant function with pieces $(x_1, y_1), \ldots, (x_p, y_p)$
  - Every time a new piece starts, objective function value increases
- **Returning a solution.** For each $i = 1, \ldots, p$ do:
  - Spend budget x_i on solution from X and budget $B - x_i$ on solution from Y using fractional knapsack
  - Fractional knapsack solution can be queried from binary search trees
    - In the analysis, we prove that removing the item which is fractionally cut in the fractional knapsack solution is not a problem





Conclusion

Making Dynamic Programming Dynamic

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- We provide a general framework such that if

  a DP has monotone rows, the dependency tree is of small height and rows are "easy to compute",

  then we can compute a $(1+\varepsilon)$-approximate solution in near-linear time and dynamically with polylog update times

- First near-linear time and dynamic algorithms for k-Balanced Graph Partitioning
- Fastest fully dynamic algorithm for Knapsack
  - Can you implement it?
- We believe there will be many applications in the future (also great for thesis topics)

0	4	5	10	17
0	0	14	29	60
0	1	1	2	5
0	16	29	29	29
0	8	12	20	22